length of the outer cylinder. Other dimensions are shown in Fig. 1.

**Configuration I**

Again, by definition[3] and referring to Fig. 1

$$L\,F_{1-3}=\int_0^L F_{d1-3}\mathrm{d}y \tag{3}$$

By introducing appropriate view factor algebra, Eq. (3) can be written

$$L\,F_{1-3}=\int_0^A (F_{d1-2}-F_{d1-4})\,\mathrm{d}y$$

$$+\int_A^{A+Y}(1-F_{d1-2}-F_{d1-4})\,\mathrm{d}y+\int_{A+Y}^{L}(F_{d1-4}-F_{d1-2})\,\mathrm{d}y \tag{4}$$

In Fig. 1 and Eq. (4), surfaces 2 and 4 are imaginary annular disks covering the ends of the outer cylinder. A term-by-term comparison of the integrations indicated in Eq. (4) with the cylinder-disk definition, Eq. (1), shows that the solution of Eq. (4) can be written immediately in terms of the cylinder-disk view factors given in Fig. 3 or by Eq. (2). After som algebraic manipulation

$$LF_{1-3}=Y+AF_A+BF_B$$

$$-(A+Y)F_{(A+Y)}-(B+Y)F_{(B+Y)} \tag{5}$$

A view factor shorthand notation is used in Eq. (5) and below where, for instance, $F_A$ is the cylinder-disk view factor $F_{1-2}$ with $\ell/r=A/r$.

**Configuration II**

$$LF_{1-3}=AF_A+a(1-F_a)$$

$$+(Y-a)F_{(Y-a)}-(A+Y)F_{(A+Y)} \tag{6}$$

**Configuration III**

$$LF_{1-3}=L+bF_b+cF_c-(L+b)F_{(L+b)}-(L+c)F_{(L+c)} \tag{7}$$

**Configuration IV**

$$LF_{1-3}=(L+D)F_{(L+D)}+(Y+D)F_{(Y+D)}$$

$$-DF_D-(L+D+Y)F_{(L+D+Y)} \tag{8}$$

## Discussion

A quick check on the validity of the previous equations can be obtained by examining some limiting cases. Configurations I and III reduce to the identical result when $A=B=b=c=0$ in which case both Eqs. (5) and (7) give $F_{1-3}=1-2\,F_L$, the equal-length concentric cylinder case. Equations (5) and (6) agree when $B=0$ and $a=Y$ as do Eqs. (6) and (8) with $a=D=0$. Also, Eq. (8) agrees with results computed in Ref. 1.

Any of the view factors given by Eqs. (5-8) can quickly be determined by reading the cylinder-disk factors from Fig. 3. Comparisons with computer programed results for several of the configurations have indicated that 2-significant-figure accuracy or better can be obtained consistently using Fig. 3 plotted on graph paper. The technique presented here is also applicable to a concentric inner cylinder and outer truncated cone with slight modification.

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# Effect of Thermal Gradient on Frequencies of Tapered Rectangular Plates

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## Nomenclature

| | |
|---|---|
| $a$ | = length of the plate |
| $b$ | = width of the plate |
| $C_i$ | = constants of linear combination |
| $\bar{D}(x)$ | = flexural rigidity variation |
| $E$ | = modulus of elasticity |
| $p$ | = circular frequency of vibration |
| $r$ | = $\pi a/b$, a parameter |
| $T$ | = temperature excess above a given reference |
| $\bar{W}(\bar{x},\bar{y},t)$, $W(x,y)$ | = lateral deflections of plate |
| $\bar{x},\bar{y}$ | = coordinates in the plane of the plate |
| $x$ | = nondimensional coordinate |
| $\alpha$ | = $\gamma T_0$, a parameter |
| $\lambda$ | = $\rho a^4 h_0 p^2/gD$, eigenvalue relating to frequency |
| $\nu$ | = Poisson's ratio |
| $\rho$ | = mass density of the material of the plate |
| $\emptyset(X)$ | = function related to plate deflection |

## Introduction

CONSIDERABLE work has been done on the vibrations of uniform and tapered rectangular isotropic plates.[1-7] It is well known,[8] that in the presence of constant thermal gradients the elastic coefficients of homogeneous materials become functions of space variables. Recently, Fauconneau and Marangoni[9] studied the effect of the nonhomogeneity caused by a thermal gradient on the natural frequencies of simply-supported plates of uniform thickness. Upper and lower bounds are computed using the Rayleigh-Ritz method and the Bazley-Fox second projection method,[10] respectively. The present investigation is to study the effect of a constant thermal gradient on the frequencies of tapered rectangular isotropic plate which is simply-supported on one pair of edges and with combinations of clamped and simply supported conditions on the other pair of edges. Design formulas for the fundamental frequency parameter are derived by using Galerkin's method. Results for uniform simply supported plates compared well with those of Ref. 9.

## Analysis

It is assumed that the tapered plate is of isotropic material subjected to a steady one-dimensional temperature

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distribution,

$$T = T_0 (1 - x) \tag{1}$$

where $T$ denotes the temperature excess above the reference temperature at any point $x$, and $T_0$ denotes the temperature excess above the reference temperature at the end $\bar{x} = a$ and $x = \bar{x}/a$. It is also assumed that the thickness variation and the maximum plate thickness are linear, and that the displacements are small compared with the wave length of vibration.

The temperature dependence of the modulus of elasticity for most engineering materials is given by

$$E(T) = E_l (1 - \gamma T) \tag{2}$$

where $E_l$ is the value of the modulus at some reference temperature.

Taking as the reference temperature, the temperature at the end of the plate $\bar{x} = a$, the modulus variation becomes

$$E(x) = E_l [1 - \alpha (1 - x)] \tag{3}$$

where

$$\alpha = \gamma T_0 \; (0 \le \alpha \le 1) \tag{4}$$

The governing equation of motion of the tapered plate is,[5]

$$\frac{\partial^2}{\partial \bar{x}^2} \left[ \bar{D} \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \nu \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} \right) \right] + \frac{\partial^2}{\partial \bar{y}^2} \left[ \bar{D} \left( \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} + \nu \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right) \right] + 2(1-\nu) \frac{\partial^2}{\partial \bar{x} \partial \bar{y}} \left( \bar{D} \frac{\partial^2 \bar{w}}{\partial \bar{x} \partial \bar{y}} \right) + \frac{\rho h}{g} \frac{\partial^2 \bar{w}}{\partial t^2} = 0 \tag{5}$$

throughout the domain of the plate.

Assuming the plate to be undergoing free vibration at frequency $p$, $\bar{w}(\bar{x}, \bar{y}, t)$ can be expressed as

$$\bar{w}(\bar{x}, \bar{y}, t) = w(\bar{x}, \bar{y}) \sin pt \tag{6}$$

Assuming further that the thickness and rigidity vary in the $\bar{x}$-direction only, Eq. (5) becomes

$$\bar{D} \left[ \frac{d^4 w}{d\bar{x}^4} + 2 \frac{d^4 w}{d\bar{x}^2 d\bar{y}^2} + \frac{d^4 w}{d\bar{y}^4} \right] + 2 \frac{d\bar{D}}{d\bar{x}} \left[ \frac{d^3 w}{d\bar{x}^3} + \frac{d^3 w}{d\bar{x} d\bar{y}^2} \right] \frac{d^2 \bar{D}}{d\bar{x}^2} \left[ \frac{d^2 w}{d\bar{x}^2} + \nu \frac{d^2 w}{d\bar{y}^2} \right] = \frac{\rho h p^2}{g} w \tag{7}$$

For simply-supported boundary conditions along $\bar{y} = 0$ and $\bar{y} = b$, a Levy type solution can be assumed for the deflection as

$$w(\bar{x}, \bar{y}) = \emptyset(\bar{x}) \sin(\pi \bar{y}/b) \tag{8}$$

In view of the previous assumption, the present analysis is restricted to modes having only a single wave along the $y$-direction. Assuming linear variation in thickness,

$$h(x) = h_o (1 + \beta x) \tag{9}$$

in which $\beta$ is the taper ratio. Considering Eqs. (3 and 9), the rigidity becomes

$$\bar{D} = D_o [1 - \alpha (1 - x)] (1 + \beta x)^5 \tag{10}$$

in which,

$$D_o = [E_l h_o^3 / 12(1 - \nu^2)] \tag{11}$$

On substitution of Eqs. (8-10) into Eq. (7), the differential equation obtained in nondimensional form is

$$\begin{aligned}
&(1+\beta x)^2 (1-\alpha+\alpha x)(d^4\emptyset/dx^4) \\
&\quad + 2(1+\beta x)[\alpha(1+\beta x) + 3\beta(1-\alpha+\alpha x)](d^3\emptyset/dx^3) \\
&\quad + 2[3\alpha\beta(1+\beta x) + 3\beta^2(1-\alpha+\alpha x) \\
&\quad - r^2(1+\beta x)^2](d^2\emptyset/dx^2) \\
&\quad - 2r^2(1+\beta x)[\alpha(1+\beta x) + 3\beta(1-\alpha+\alpha x)](d\emptyset/dx) \\
&\quad + r^2[r^2(1+\beta x)^2 - 6\nu\alpha\beta(1+\beta x) \\
&\quad - 6\nu\beta^2(1-\alpha+\alpha x)]\emptyset = \lambda\emptyset
\end{aligned} \tag{12}$$

For the cases considered herein, the boundary conditions at $x = 0$ and $x = 1$ are given in the following.

Simply supported end

$$\emptyset = 0, \quad (d^2\emptyset/dx^2) = 0 \tag{13}$$

Clamped end

$$\emptyset = 0, \quad (d\emptyset/dx) = 0 \tag{14}$$

Equation (12) is solved in conjunction with combinations of the boundary conditions of Eqs. (13 and 14) by using the well-known Galerkin technique and the design formulas for the fundamental frequency are derived.

## Frequency Equations

For the development of a solution by the Galerkin technique, the plate is assumed to have a deflection of the form

$$\emptyset(x) = \sum_{i=0}^{4} -C_i x^i \tag{15}$$

in which the constants $C_i$ $(i = 0,1,2,3,4)$ can be obtained as ratios of the constant $C_4$ using the combinations of the boundary conditions of Eqs. (13 and 14).

The assumed functions $\emptyset(x)$ satisfying the boundary conditions, and the resulting frequency equations giving the fundamental frequency of vibration for the four cases considered herein are given in the following:

a) Plate simply-supported along $x = 0$ and $x = 1$:

$$\emptyset(x) = C_4 (x - 2x^3 + x^4) \tag{16}$$

$$\begin{aligned}
\lambda = {} & (97.54843 - 48.77421\alpha + 97.54840\beta - 52.25790\,\alpha\beta \\
& - 0.58085\,\alpha\beta^2 - 2.32247\,\beta^2) \\
& + r^2(19.74192 - 9.87094\alpha + 19.74196\beta \\
& - 3.51616\,\alpha\beta - 4.90326\,\alpha\beta^2 + 8.61290\,\beta^2) \\
& + r^4(1 - 0.49999\,\alpha + \beta - 0.43401\,\alpha\beta \\
& - 0.10850\,\alpha\beta^2 + 0.28299\,\beta^2) \\
& - \nu r^2(6\,\alpha\beta + 0.00001\,\alpha\beta^2 + 6\,\beta^2)
\end{aligned} \tag{17}$$

b) Plate clamped along $x=0$ and $x=1$:

$$\emptyset(x)=C_4(x^2-2x^3+x^4) \tag{18}$$

$$\begin{aligned}\lambda= {} & (503.99523-251.99603\ \alpha+503.99932\ \beta \\ & -108.00405\ \alpha\beta-18.00136\ \alpha\beta^2-179.99737\ \beta^2) \\ & +r^2(24.00005-12\alpha+24.00024\ \beta-7.00063\ \alpha\beta \\ & -17.75058\ \alpha\beta^2+8.99981\ \beta^2) \\ & +r^4(1-0.49998\ \alpha+\beta-0.45460\ \alpha\beta-0.11361\ \alpha\beta^2 \\ & +0.27270\ \beta^2)-\nu r^2(6\alpha\beta+0.00012\ \alpha\beta^2+6\beta^2)\end{aligned} \tag{19}$$

c) Plate simply supported along minimum thickness edge $x=0$, and clamped along maximum thickness edge $x=1$:

$$\emptyset(x)=(C_4/2)\ (x-3x^3+2x^4) \tag{20}$$

$$\begin{aligned}\lambda= {} & (238.73721-99.47369\ \alpha+311.68499\ \beta \\ & -95.68475\ \alpha\beta-5.68378\ \alpha\beta^2+90.94744\ \beta^2) \\ & +r^2(22.73688-14.21060\ \alpha+17.05257\ \beta-5.10506\ \alpha\beta \\ & -7.26333\ \alpha\beta^2+6.47376\ \beta^2) \\ & +r^4(1-0.56578\ \alpha+0.86842\ \beta-0.43780\ \alpha\beta \\ & -0.09808\ \alpha\beta^2+0.21531\ \beta^2) \\ & -\nu r^2(6\alpha\beta-0.78944\ \alpha\beta^2+6\beta^2)\end{aligned} \tag{21}$$

d) Plate clamped along minimum thickness edge $x=0$, and simply supported along maximum thickness edge $x=1$:

$$\emptyset(x)=(C_4/2)\ (3x^2-5x^3+2x^4) \tag{22}$$

$$\begin{aligned}\lambda= {} & (238.73971-139.26480\alpha+165.79168\beta \\ & -62.52888\alpha\beta-8.52350\alpha\beta^2+17.99903\beta^2) \\ & +r^2(22.73723-8.52680\alpha+28.42080\beta-5.10426\alpha\beta \\ & -13.07961\alpha\beta^2+12.15831\beta^2) \\ & +r^4(1-0.43419\alpha+1.1316\beta-0.43782\alpha\beta \\ & -0.12080\alpha\beta^2+0.34688\beta^2) \\ & -\nu r^2(6\alpha\beta+0.78963\alpha\beta^2+6\beta^2)\end{aligned} \tag{23}$$

When $\alpha=0$, the previous frequency equations reduce to those for tapered plates neglecting the effect of temperature gradient. When $\beta=0$, the previous frequency equations reduce to those for uniform plates including the effect of temperature gradient. When $\alpha=0$ and $\beta=0$, the previous frequency equations reduce to those for uniform plates neglecting the effect of temperature gradient.

## Results and Conclusions

Table 1 shows the comparison between the results for $\lambda$ for $\alpha=0.5$ given in Ref. 8 and those obtained from Eq. (17) for uniform ($\beta=0$) simply-supported plates. It can be observed that even out of so much of approximation in the present analysis the results compared favorably well and are well within engineering accuracy.

Results for the fundamental frequency parameter $\lambda$, for a square plate ($a/b=1$) for which Poisson's ratio $\gamma=0.3$, ob-

**Table 1 Comparison of results from Eq. (17) for uniform simply-supported plates with the upper bounds from 15 × 15 matrix of Ref. 8 for $\alpha=0.5$.**

| Aspect ratio $a/b$ | Fundamental frequency parameter $\lambda$: Present analysis Eq. (17) | Upper bounds[8] | Percentage deviation |
|---|---|---|---|
| 0.5 | 114.298 | 112.298 | +1.78 |
| 1.0 | 292.587 | 287.068 | +1.92 |
| 1.5 | 772.673 | 755.980 | +2.20 |
| 2.0 | 1828.959 | 1782.021 | +2.63 |

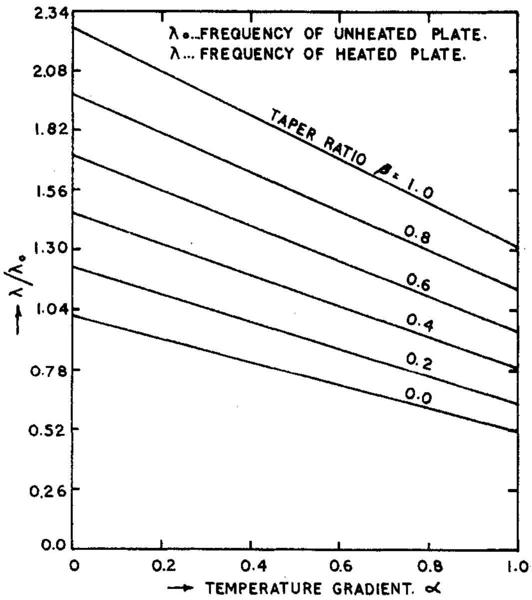


**Fig. 1 Effect of temperature gradient on the fundamental frequency of a tapered square plate simply-supported along $\bar{y}=0$ and $\bar{y}=b$ and $\bar{x}=0$ and $\bar{x}=a$.**

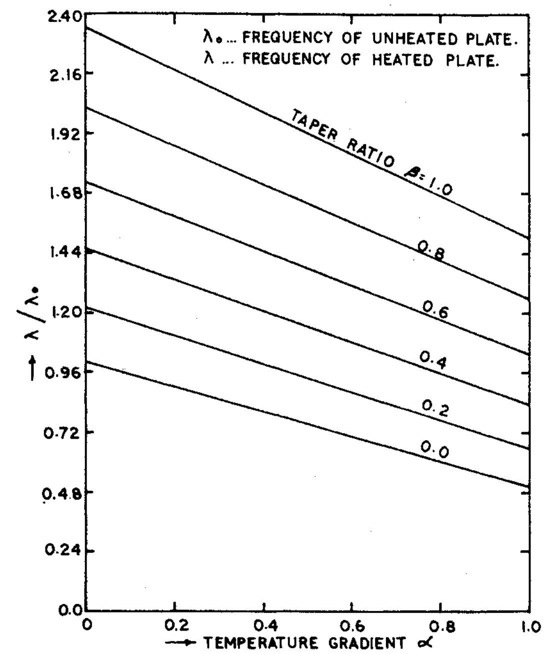


**Fig. 2 Effect of temperature gradient on the fundamental frequency of a tapered square plate simply-supported along $\bar{y}=0$ and $\bar{y}=b$ and clamped along $\bar{x}=0$ and $\bar{x}=a$.**

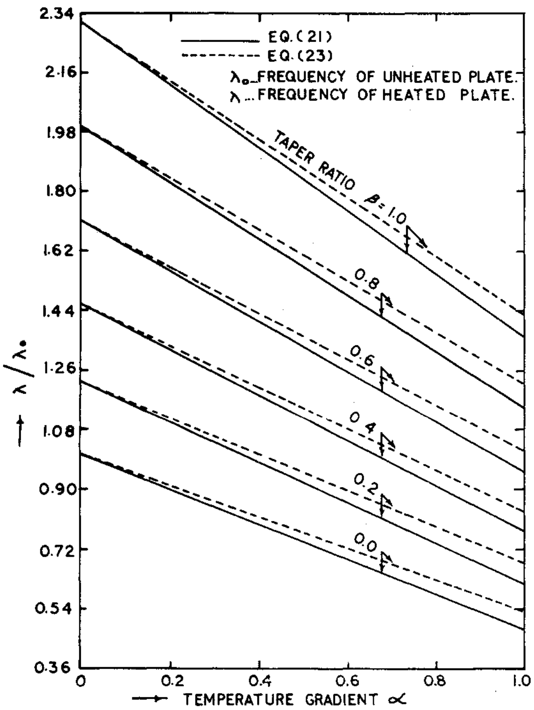


**Fig. 3 Effect of temperature gradient on the fundamental frequency of a tapered square plate simply-supported along $\bar{y}=0$ and $\bar{y}=b$ and with cases c) and d) along $\bar{x}=0$ and $\bar{x}=a$.**

tained from Eqs. (17) and (19) for cases a) and b), are plotted in Figs. 1 and 2, respectively, for various values of temperature gradient parameter $\alpha$ and thickness taper parameter $\beta$. Results for $\lambda$, for a square plate, obtained from Eqs. (21) and (23) for the cases c) and d) are plotted in Fig. 3, for various values of $\alpha$ and $\beta$ [thick lines for case c) and dotted lines for case d)]. It can be observed from Figs. 1-3 that in all the cases considered herein, for any specific value of $\beta$, the frequency parameter $\lambda$ decreases for increasing values of $\alpha$. The variation of $\lambda$ for increasing values of $\alpha$ becomes steeper for increasing values of taper parameter $\beta$. From Fig. 3, it can be seen that the values of the fundamental frequency parameter $\lambda$ are higher for case d) than those for case c), and that this difference increases for increasing values of $\alpha$ for any specific value of $\beta$.

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